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Generalized Konishi anomaly, Seiberg duality and singular effective superpotentials

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Abstract: Using the generalized Konishi anomaly (GKA) equations, we derive the effective superpotential of four-dimensional $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N_c)$ gauge theory with $N_f = N_c + 2$ fundamental flavors. We find, however, that the GKA equations are only integrable in the Seiberg dual description of the theory, but not in the direct description of the theory. The failure of integrability in the direct, strongly coupled, description suggests the existence of non-perturbative corrections to the GKA equations.

Keywords: Supersymmetric gauge theory, Supersymmetric Effective Theories, Supersymmetry and Duality.

Contents

1. Introduction

Holomorphicity of the superpotential and gauge couplings, global symmetries and the weak-coupling limit enable one to obtain exact results in supersymmetric gauge theories (for reviews see [1, 2]), making these theories more tractable than their non-supersymmetric cousins. Since supersymmetric gauge theories exhibit a wealth of non-perturbative phenomena such as dynamically generated superpotentials with associated confinement or chiral symmetry breaking [3, 4], deformed classical moduli spaces [5], Seiberg duality [6], etc.., and since some of these phenomena also occur in non-supersymmetric theories, supersymmetric gauge theories are usually considered as a way to qualitatively study non-perturbative aspects of ordinary gauge theories. Therefore having a clear picture of the behavior of supersymmetric gauge theories may shed light on a better understanding of the dynamics of strongly-coupled gauge theories with no supersymmetry.

Despite much progress in the effective dynamics of four-dimensional $\mathcal{N} = 1$ supersymmetric QCD, the behavior of the effective superpotential for a number of flavors N_f large compared to the number of colors N_c is not well-understood. This is because, firstly, when the number of flavors increases there are typically additional light degrees of freedom at the origin of the moduli space that one needs to include in the effective description. Secondly, the effective superpotentials become singular when expressed in terms of the local gauge-invariant light degrees of freedom away from the origin; more precisely, the potentials derived from such effective superpotentials have cusp-like singularities at their minima [7]. Thirdly, the dependence of these effective superpotentials on the strong coupling scale of the theory Λ is such that they apparently diverge in the the weak coupling limit $\Lambda \to 0$. Because of these problems the physical meaning of such superpotentials is thought to be problematic.

We have argued elsewhere [7] that effective superpotentials for the light gauge-invariant degrees of freedom away from the origin must nevertheless exist. Furthermore, direct computation in the case of SU(2) superQCD shows [7] that these superpotentials, although singular, are nevertheless physically sensible, and reproduce both the low energy physics as well as certain higher-derivative terms in an intrinsic description on the moduli space away from the origin [8].

In this paper we extend our arguments to $\mathrm{SU}(N_c)$ superQCD by computing its singular effective superpotential. Unlike the SU(2) case, the $\mathrm{SU}(N_c)$ case has a smaller global symmetry group, making it harder to find the superpotential. We deal with this by solving a system of differential equations for the effective superpotential [9] derived from the generalized Konishi anomaly (GKA) equations [10]. The complexity of this system increases with the number of massless fundamental flavors N_f, but we are able to solve them in the first interesting case, $N_f = N_c + 2$.

However, there are some subtleties involved in applying the GKA equations in this case: the GKA equations are not integrable when applied to the (strongly coupled) direct description of the theory, but are integrable when applied to the (weakly coupled) Seiberg dual description of the theory. In the rest of this section, we will explain these subtleties in more detail and discuss the issues that they raise concerning the possible non-perturbative exactness of the GKA equations. We leave the technical details of the calculations to subsequent sections.

The indirect argument for the existence of the effective superpotential referred to above goes as follows: Wilsonian effective superpotentials are assured to exist only if there is a region in the configuration space of the chosen chiral fields where all of them are light together and comprise all the light degrees of freedom. If this condition is satisfied, then the resulting effective superpotential can be extended over the whole configuration space by analytic continuation using the holomorphicity of the superpotential. For a large enough number of flavors the theory becomes IR free and the only region where all the components of the chosen chiral vevs become light at the same time is at the origin. We know what the light degrees of freedom are near that point since we have a weakly coupled lagrangian description there. The physics can be made arbitrarily weakly coupled simply by taking all scalar field vevs $\langle\phi\rangle << \Lambda$ where Λ is the strong coupling scale (or UV cutoff) of the IR free theory. In this limit the physics is just the classical Higgs mechanism, and all particles get masses of order $\langle\phi\rangle$ or less. The Wilsonian effective description results from integrating out modes with energies greater than a cutoff, which we take to be some multiple of $\langle\phi\rangle$. The effective action will then include all local gauge-invariant operators made from the fundamental fields in the lagrangian and which can create particle states with masses below the cutoff. For the purpose of constructing the effective superpotential, the relevant local gauge-invariant operators are those in the chiral ring. It is then just a matter of constructing in the classical gauge theory a set of operators which generate the chiral ring. We will refer to this set as the classical chiral operators of the theory.

In a weakly coupled $\mathrm{SU}(N_c)$ superQCD a basis of local gauge-invariant operators in the chiral ring (the classical chiral operators) is comprised of just the glueball, meson, and baryon operators [10, 11]. An effective superpotential which is a function of these operators must then exist. For $N_f > N_c+1$, the quantum moduli space is also the same as the classical one [5], but effective superpotentials (singular or not) for these cases have not been found before. Also, this theory has an equivalent description in the IR in terms of a "Seiberg dual" SU(N_f-N_c) supersymmetric QCD with N_f (dual) fundamental quarks and anti-quarks and a set of singlet scalars coupled to the dual mesons through a superpotential [6].

Note that the above argument does not directly show the existence of such effective superpotentials in the asymptotically free case. In particular, for theories in the "conformal window" where neither the direct nor Seiberg dual description is IR free ($\frac{3}{2}N_c < N_f < 3N_c$ for SU(N_c) gauge group), we have no useful description of the light degrees of freedom at the origin of moduli space. Nevertheless, given an effective superpotential for an IR free theory, one can then successively add mass terms to the effective superpotential and integrate out massive flavors to derive consistent effective superpotentials in the conformal window. This then assures us that effective superpotentials exist for all numbers of light flavors in supersymmetric QCD.

Our method for deriving the effective superpotential for the $N_f > N_c+1$ theory will be to integrate the generalized Konishi anomaly (GKA) equations [10] following the approach of [12, 9]. The resulting equations become very complicated [9] for large numbers of flavors, so we are not able to solve them directly in the IR free case $N_f \geq 3N_c$, and then integrate out flavors as in the above argument.

For $N_f = N_c + 2$, however, the GKA equations simplify to a first order matrix differential equation simple enough that we can analyze it. We show that the GKA equations for the effective superpotential are not integrable in this case for $N_c > 2$. This is not in direct conflict with the general arguments advanced above: for $N_f = N_c + 2$ and $N_c > 2$, the theory at the origin is strongly coupled in terms of its microscopic fields, so an effective description in terms of the chiral ring operators made from these fields simply need not exist.

However, this failure of integrability presents a sharper puzzle in light of the following: we can nevertheless calculate an effective superpotential by using the fact that for $N_f = N_c + 2$ with $N_c \geq 4$ the Seiberg dual description is an IR free SU(2) gauge theory. By applying the GKA equations to the Seiberg dual description we derive the effective superpotential of the theory in terms of the dual chiral fields. It is given in equation (4.18) below, where we have used the map [6] between direct and dual chiral operators to interpret this as an effective superpotential in terms of the classical chiral operators of the direct theory — the mesons M^i_j, baryons B_{ij} and $\tilde{B}^{ij}$, and the glueball S.

We have thus found the effective superpotential in terms of the classical chiral operators. This raises the question of why were the GKA equations not integrable in the direct theory in the first place? We interpret this failure of integrability as indicating that the GKA equations get non-perturbative corrections. It remains to characterize more precisely the nature of the non-perturbative corrections to the GKA equations. One possibility is that there exists a non-perturbatively modified set of GKA equations in terms of the classical chiral operators (i.e., glueball, mesons, and baryons in our case). Another possibility is that there are additional operators in the chiral ring which are independent of the classical chiral operators; when included, they could render the GKA equations integrable, in much the same way that the extra singlet field in the Seiberg dual description does. It has not been ruled out that such additional fields could be seen in the semi-classical description as higher derivative chiral operators.[1] (Note that the non-trivial higher-derivative chiral operators constructed in [8] are *not* candidates, since they are $\overline{Q}$-exact when extended off the moduli space to the configuration space of chiral vevs.) Finally, both possibilities — a non-perturbative deformation of the GKA equations and the inclusion of additional chiral fields — could occur together.

Once the issue of non-perturbative corrections to the GKA equations is raised, it applies equally well to the GKA equations derived in the Seiberg dual description. It is an open question whether the effective superpotential derived below in the dual description is correct or not for $N_f \geq 6$. For even though, when $N_f \geq 6$, the dual description is weakly coupled at the origin of moduli space, it becomes strongly coupled an arbitrarily small distance away from the origin since the superpotential term in the dual theory destabilizes the free fixed point at the origin [6].

The remainder of the paper carries out the computations described qualitatively above, and is organized as follows. To illustrate the method in a simple case first, and for later comparison to the Seiberg dual description of the $\mathrm{SU}(N_c)$ case, in section 2 we consider the case of SU(2) gauge group with $N_f \geq 4$ flavors. We show how the GKA equations written in terms of SU(2) mesons and baryons gives an effective superpotential matching that found in [7] where we worked instead with the single antisymmetric meson field appropriate to an Sp(1) description of the theory (i.e., one which makes the enlarged global symmetry group of the SU(2) compared to the general $\mathrm{SU}(N_c)$ theory manifest). In section 3 we apply the GKA equations to $\mathrm{SU}(N_c)$ superQCD with $N_f = N_c + 2$ and $N_c \geq 4$. We show that the resulting equations for the effective superpotential are not integrable for $N_f \geq 6$, but that they are integrable and match the SU(2) result of the previous section for $N_f = 4$.

In section 4 we apply the GKA equations to the Seiberg dual of the theory in section 3, and solve for the effective superpotential. The form of this superpotential is complicated: integrating out the heavy glueball gives an effective superpotential of the form $\sqrt{\det M} f(X)$ where $X = M\tilde{B}M^T B/\det M$, but a closed-form expression for $f(X)$ is not found. Instead, we show that f obeys a nonlinear first order matrix differential equation (4.34). A power series expansion of the solution to order X^4 is computed in (4.29). We then compare this result to the SU(2) effective superpotential of section 2 when $N_f = 4$, and show that they agree at least to order X^4.

2. $\mathcal{W}_{\mathrm{eff}}$ for $\mathbf{N_f \geq 4}$ SU(2) superQCD

We show how to calculate the effective superpotential of SU(2) superQCD with $N_f \geq 4$ using the generalized Konishi anomaly equations, following [12, 9].

SU(2) superQCD has N_f massless quark and anti-quark chiral multiplets, Q^i_a and $\tilde{Q}^a_i$, transforming in the $\mathbf{2}$ and $\overline{\mathbf{2}}$ of the gauge group, respectively. Here $a = 1, 2$ is the color index and $i = 1, \ldots, N_f$ the flavor index. The apparent global symmetry of the theory is

[1] We thank S. Hellerman for discussions on this point.

$\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$. Since $\mathbf{2}$ and $\overline{\mathbf{2}}$ are equivalent representations, though, SU(2) superQCD actually has the larger symmetry group $\mathrm{SU}(2N_f) \times \mathrm{U}(1)_R$, which is large enough to determine the effective superpotential uniquely [7]. Here, since we are looking to the generalization to $\mathrm{SU}(N_c)$ which only has the smaller symmetry group, we will analyze the SU(2) case in terms of the Q's and $\tilde{Q}$'s keeping only the $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$ symmetry manifest.

The classical moduli space is parameterized by the vevs of the meson M and the baryon $\mathcal{B}$, $\tilde{\mathcal{B}}$ chiral superfields defined by

$$M^i{}_j := Q^i_a \tilde{Q}^a_j, \qquad \mathcal{B}^{ij} := \epsilon^{ab} Q^i_a Q^j_b, \qquad \tilde{\mathcal{B}}_{ij} := \epsilon_{ab} \tilde{Q}^a_i \tilde{Q}^b_j, \qquad S := \mathrm{tr}(W^\alpha W_\alpha)/(32\pi^2), \tag{2.1}$$

where we have also defined the glueball chiral superfield S. These fields can be assigned the charges $R(S) = 2$ and $R(M) = R(B) = R(\tilde{B}) = 2(N_f - 2)/N_f$ under the non-anomalous $\mathrm{U}(1)_R$ symmetry. The meson and baryon vevs cannot take arbitrary values but are subject to constraints following from (2.1),

$$\mathcal{B}^{[ij} M^{k]}_{\ell} = M^i{}_{[j} \tilde{\mathcal{B}}_{k\ell]} = \mathcal{B}^{[ij} \mathcal{B}^{k]\ell} = \tilde{\mathcal{B}}_{i[j} \tilde{\mathcal{B}}_{k\ell]} = M^{[i}_{\ \,k} M^{j]}_{\ \,\ell} - \mathcal{B}^{ij} \tilde{\mathcal{B}}_{k\ell} = 0, \tag{2.2}$$

where the square brackets denote antisymmetrization. These constraints imply that M, $\mathcal{B}$, and $\tilde{\mathcal{B}}$ all have rank less than or equal to 2 and, up to flavor rotations, take the form

$$M = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & \mathbf{0} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} & b & \\ -b & & \\ & & \mathbf{0} \end{pmatrix}, \qquad \tilde{\mathcal{B}} = \begin{pmatrix} & \widetilde{b} & \\ -\widetilde{b} & & \\ & & \mathbf{0} \end{pmatrix}, \tag{2.3}$$

with $m_1 m_2 = b\widetilde{b}$ and $\mathbf{0}$ the $(N_f - 2) \times (N_f - 2)$ matrix of zeros. For $N_f \geq 3$, the quantum moduli space is also the same as the classical one [5].

For SU(2) superQCD with fundamental flavors M, $\mathcal{B}$, $\tilde{\mathcal{B}}$, and S are thought to generate all non-trivial local gauge-invariant operators in the chiral ring of the classical theory [10, 11]. When $N_f = 4$ or 5 the theory is strongly coupled, and has new massless degrees of freedom at the origin of moduli space, so the chiral ring might be deformed or enlarged from the classical answer. But for $N_f \geq 6$, where the theory is IR free, the classical description is as accurate as we like (in the vicinity of the origin of the moduli space). So we will make the assumption that we can write our effective superpotential in terms of just S, M, $\mathcal{B}$, and $\tilde{\mathcal{B}}$.

However, for $N_f \geq 4$, the global symmetries allow infinitely many terms in the effective superpotential, making it hard to guess its correct form. So, instead, we use the generalized Konishi anomaly (GKA) equations to derive the effective superpotential. If $F^i_r(\Phi, W_\alpha)$ are holomorphic functions transforming in the same representation of the gauge group as a chiral superfield Φ^i_r (i is a flavor index and r an index for the gauge representation), then the GKA equation [10] is

$$\left\langle \frac{\partial \mathcal{W}_{\mathrm{tree}}}{\partial \Phi^j_r} F^i_r \right\rangle = \frac{1}{32\pi^2} \left\langle (W^\alpha W_\alpha)^s_t \frac{\partial F^i_s}{\partial \Phi^j_t} \right\rangle, \tag{2.4}$$

which can be interpreted as the anomalous Ward identity coming from the field transformation $\delta\Phi^i_r = F^i_r$. Here $\mathcal{W}_{\text{tree}}$ is the classical superpotential. The GKA equation is perturbatively one-loop exact [10]. It has also been shown [13] that it does not get non-perturbative corrections for a U(N) gauge theory with matter in the adjoint representation as well as for Sp(N) and SO(N) gauge theories with matter in symmetric or antisymmetric representations. For the theories we are discussing here, its non-perturbative status is not known; however, as we show below, there is strong evidence, at least for SU(2), that the GKA equations are actually non-perturbatively exact.

Consider now SU(2) superQCD with the classical superpotential

$$\mathcal{W}_{\text{tree}} = m^i{}_j(\hat{M}^j{}_i - M^j{}_i) + b_{ij}(\hat{\mathcal{B}}^{ij} - \mathcal{B}^{ij}) + \widetilde{b}^{ij}(\hat{\tilde{\mathcal{B}}}_{ij} - \tilde{\mathcal{B}}_{ij}). \tag{2.5}$$

Here $m^i{}_j, b_{ij}$ and $\widetilde{b}^{ij}$ are Lagrange multipliers constraining the operators $\hat{M}^j{}_i$, $\hat{\mathcal{B}}^{ij}$ and $\hat{\tilde{\mathcal{B}}}_{ij}$ to have $M^j{}_i$, $\mathcal{B}^{ij}$ and $\tilde{\mathcal{B}}_{ij}$ as their vacuum expectation values, respectively. (Whenever we need to distinguish an operator from its vev, we put a hat on the operator.) We are looking for the effective superpotential $\mathcal{W}_{\text{eff}}$ as a function of the vevs S, M, $\mathcal{B}$, and $\tilde{\mathcal{B}}$. It follows from (2.5) and the nature of the Legendre transform [14, 15, 1] that

$$m^i{}_j = -\frac{\partial \mathcal{W}_{\text{eff}}}{\partial M^j{}_i}, \qquad b_{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \mathcal{B}^{ij}}, \qquad \widetilde{b}^{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \tilde{\mathcal{B}}_{ij}}, \tag{2.6}$$

where the factors of 2 come from the antisymmetry of the baryons.

We now use the GKA equations to determine the dependence of the Lagrange multipliers on M, $\mathcal{B}$, $\tilde{\mathcal{B}}$, and S. First set $F^i_r = \Phi^i_r = Q^i_a$ in (2.4) yielding

$$Mm = S + 2\mathcal{B}b, \tag{2.7}$$

where we are using a matrix notation on the flavor indices (so that, e.g., the last equation stands for $M^i{}_k m^k{}_j = S\delta^i{}_j + 2\mathcal{B}^{ik}b_{kj}$). A similar equation,

$$mM = S + 2\widetilde{b}\tilde{\mathcal{B}}, \tag{2.8}$$

follows from taking $F^i_r = \Phi^i_r = \tilde{Q}^a_i$. Two more independent equations follow from taking $F^i_r = \epsilon_{ab}\tilde{Q}^b_i$ and $\Phi^i_r = Q^i_a$, and from taking $F^i_r = \epsilon^{ab}Q^i_b$ and $\Phi^i_r = \tilde{Q}^a_i$, giving

$$\tilde{\mathcal{B}}m = -2M^T b, \qquad m\mathcal{B} = -2\widetilde{b}M^T. \tag{2.9}$$

We will carry out subsequent calculations at a generic point on the configuration space of S, M, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ where they are all invertible matrices. Note, however, that when N_f is odd, $\mathcal{B}$ and $\tilde{\mathcal{B}}$, being odd rank antisymmetric matrices, are never invertible. We get around this problem by restricting ourselves to an even number of flavors only. Once we have found the superpotential for even N_f's, we can add a mass term for one flavor and integrate it out to get the effective superpotential for the odd $N_f - 1$ flavors.

Therefore, multiplying (2.7–2.9) by appropriate inverses and substituting for the Lagrange multipliers m, b, and $\widetilde{b}$ using (2.6) gives a set of partial differential equations for

$\mathcal{W}_{\text{eff}}$

$$\begin{aligned}
\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \mathcal{B}^{ij}} &= S\left[(\tilde{\mathcal{B}}M^{-1}\mathcal{B}+M^T)^{-1}\tilde{\mathcal{B}}M^{-1}\right]_{ij}, \\
\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \tilde{\mathcal{B}}_{ij}} &= S\left[M^{-1}\mathcal{B}(\tilde{\mathcal{B}}M^{-1}\mathcal{B}+M^T)^{-1}\right]^{ij}, \\
\frac{\partial \mathcal{W}_{\text{eff}}}{\partial M^j_{\ i}} &= S\left[M^{-1}\mathcal{B}(\tilde{\mathcal{B}}M^{-1}\mathcal{B}+M^T)^{-1}\tilde{\mathcal{B}}M^{-1}-M^{-1}\right]^i_{\ j}.
\end{aligned} \tag{2.10}$$

Integrate the first equation in (2.10) to find

$$\mathcal{W}_{\text{eff}} = -\frac{S}{2}\ln\det\left(\mathbb{I}+M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B}\right)+G(M,\tilde{\mathcal{B}},S), \tag{2.11}$$

where $M^{-T} = (M^T)^{-1}$, $\mathbb{I}$ is the $N_f \times N_f$ identity matrix, and G is an undetermined integration function. Comparing the second equation in (2.10) with the derivative of (2.11) with respect to $\tilde{\mathcal{B}}$ gives $\partial G/\partial\tilde{\mathcal{B}} = 0$. Also, comparing the derivative of (2.11) with respect to M to the third equation in (2.10) gives $\partial G/\partial M = -SM^{-1}$, so that

$$G = -S\ln\det(M/\Lambda^2)+H(S), \tag{2.12}$$

for some undetermined function $H(S)$. The Λ-dependence was determined by dimensional analysis, where Λ is the strong-coupling scale of the SU(2) superQCD.

Equivalently, the global flavor symmetry implies that $\mathcal{W}_{\text{eff}} = \mathcal{W}_{\text{eff}}(X,\det M,S)$ where $X := M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B}$. Plugging this functional form into (2.6–2.9) gives simple matrix differential equations leading to (2.11–2.12).

$H(S)$ is determined up to a constant by the $\mathrm{U}(1)_R$ symmetry. Since $R(\mathcal{W}_{\text{eff}}) = 2$, H must be linear in S, plus a logarithmic piece to cancel the $\mathrm{U}(1)_R$ transformation of the $-S\ln\det M$ term, giving

$$H(S) = (2-N_f)S[\alpha-\ln(S/\Lambda^3)] \tag{2.13}$$

for some undetermined constant α. We can determine α by matching to the Veneziano-Yankielowicz superpotential [16], $W_{\text{VY}}(S) = 2S[1-\ln(S/\Lambda^3_{\text{YM}})]$, for pure SU(2) superYang-Mills. It is a short exercise to integrate out the mesons and baryons in (2.11) and match strong coupling scales to find $\alpha = 1$. We therefore find that the effective superpotential is

$$\mathcal{W}_{\text{eff}} = -\frac{S}{2}\ln\left[(\det M)^2\det(\mathbb{I}+M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B})\right]+(2-N_f)S(1-\ln S)+(6-N_f)S\ln\Lambda. \tag{2.14}$$

Since S is massive we can integrate it out by solving its equation of motion, $\partial\mathcal{W}_{\text{eff}}/\partial S = 0$ to find

$$\mathcal{W}_{\text{eff}}(M,\mathcal{B},\tilde{\mathcal{B}}) = (2-N_f)\left[\Lambda^{N_f-6}\det M\sqrt{\det(\mathbb{I}+M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B})}\right]^{1/(N_f-2)}. \tag{2.15}$$

This superpotential reproduces all known low energy aspects of SU(2) superQCD. The easiest way to see this is to convert it to a description which makes the full global flavor symmetry manifest. As we mentioned earlier, SU(2) superQCD with N_f fundamental Q^i_a and N_f anti-fundamental $\tilde{Q}^a_i$ can be equivalently described in terms of $2N_f$ doublets $\mathcal{Q}^I_a$, $I = 1, \cdots, 2N_f$ with $\mathcal{Q}^i_a = Q^i_a$ and $\mathcal{Q}^{N_f+i}_a = \epsilon_{ab}\tilde{Q}^b_i$. Hence M, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are combined into an antisymmetric $2N_f \times 2N_f$ matrix $V^{IJ} = \epsilon^{ab}\mathcal{Q}^I_a\mathcal{Q}^J_b$,

$$V = \begin{pmatrix} \mathcal{B} & M \\ -M^T & \tilde{\mathcal{B}} \end{pmatrix}. \tag{2.16}$$

After a bit of algebra[2] it is seen that our singular effective superpotential (2.15) can be written in terms of this new variable as

$$\mathcal{W}_{\text{eff}} = (2 - N_f)\left(\Lambda^{N_f-6}\sqrt{\det V}\right)^{1/(N_f-2)}, \tag{2.17}$$

making the SU($2N_f$) global symmetry manifest. Indeed, (2.17) coincides with the singular effective superpotential found in [7], and so it satisfies all the checks discussed there: it gives rise to the correct moduli space, is consistent under integrating out flavors, and reproduces all the higher-derivative F-terms found in [8].

The success of this calculation can be taken as evidence that the GKA equations are non-perturbatively exact for SU(2) superQCD.

3. $\mathcal{W}_{\text{eff}}$ for $\mathbf{N_f = N_c + 2}$ superQCD: non-integrability of GKA equations

SU(N_c) superQCD has N_f massless quark chiral fields Q^i_a and N_f massless anti-quark chiral fields $\tilde{Q}^a_i$ transforming in the fundamental and anti-fundamental representations, respectively. Here $i = 1, \ldots, N_f$ is the flavor index and $a = 1, \ldots, N_c$ is the color index. When $N_f = N_c + 2$ the classical moduli space is parameterized by the gauge-invariant vevs of the glueball, meson, and baryons defined by

$$\begin{aligned}
\hat{S} &:= \frac{1}{32\pi^2}\,\mathrm{tr}(W^\alpha W_\alpha), \\
\hat{M}^i_j &:= Q^i_a\tilde{Q}^a_j, \\
\hat{B}_{ij} &:= \frac{1}{N_c!}\epsilon_{ijk_1\cdots k_{N_c}}\epsilon^{a_1\cdots a_{N_c}}Q^{k_1}_{a_1}\cdots Q^{k_{N_c}}_{a_{N_c}}, \\
\hat{\tilde{B}}^{ij} &:= \frac{1}{N_c!}\epsilon^{ijk_1\cdots k_{N_c}}\epsilon_{a_1\cdots a_{N_c}}\tilde{Q}^{a_1}_{k_1}\cdots\tilde{Q}^{a_{N_c}}_{k_{N_c}}.
\end{aligned} \tag{3.1}$$

The global symmetry of the theory is SU(N_f) $\times$ SU(N_f) $\times$ U(1)$_B$ $\times$ U(1)$_R$. The U(1)$_R$ charges are $R(S) = 2$, $R(M) = 4/N_f$, and $R(B) = R(\tilde{B}) = 2N_c/N_f$. The classical moduli space is described by the constraints that M, B, and $\tilde{B}$ satisfy by virtue of their definitions,

$$B_{ik}M^k_j = M^i_k\tilde{B}^{kj} = B_{[ij}B_{k]\ell} = \tilde{B}^{[ij}\tilde{B}^{k]\ell} = \tilde{B}^{ij}B_{k\ell} - M^{-1}{}^{[i}_{k}M^{-1}{}^{j]}_{\ell}\,\det M = 0. \tag{3.2}$$

Square brackets denote antisymmetrization; antisymmetrization on n indices consists of $n!$ terms (i.e., with out a factor of $1/n!$).

[2] Write $V = \left(\begin{smallmatrix} \mathcal{B} & 0 \\ 0 & \tilde{\mathcal{B}} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ y & 1 \end{smallmatrix}\right)$ with $x := \mathcal{B}^{-1}M$, $y := -\tilde{\mathcal{B}}^{-1}M^T$. Use the identity $\det\left(\begin{smallmatrix} 1 & x \\ y & 1 \end{smallmatrix}\right) = \det(1 - xy)$, so $\det V = \det\mathcal{B}\det\tilde{\mathcal{B}}\det(1 - xy) = (\det M)^2\det(-y^{-1}x^{-1} + 1)$, which gives (2.15).

They imply that by appropriate flavor rotations M, B, and $\tilde{B}$ can be put in the form

$$M = \begin{pmatrix} \mathbf{m} & \\ & \end{pmatrix}, \qquad B = \begin{pmatrix} & & \\ & & b \\ & -b & \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} & & \\ & & \widetilde{b} \\ & -\widetilde{b} & \end{pmatrix}, \tag{3.3}$$

where $\mathbf{m}$ is an $N_c \times N_c$ matrix and b, $\widetilde{b}$ are numbers satisfying $b\widetilde{b} = \det(\mathbf{m})$. The classical and the quantum moduli spaces are the same [5], but at the origin there are extra light degrees of freedom. At points away from the origin, the only light degrees of freedom are components of M, B, and $\tilde{B}$. At the origin, $\mathrm{SU}(N_c)$ supersymmetric QCD with $N_f = N_c{+}2$ is an interacting superconformal field theory for $N_c = 2$ and 3. For $N_c \geq 4$ it becomes strongly coupled, but has an IR free dual description in the IR [6].

In this section we will try to construct an effective superpotential in terms of these fields which correctly describes the moduli space of vacua for points away from the origin. As in the last section, we will use the GKA equations to systematically derive $\mathcal{W}_{\text{eff}}$. In fact, the GKA equations were used in [9] to construct a set of coupled partial differential equations for the effective superpotentials of $\mathrm{SU}(N_c)$ supersymmetric QCD. They have been integrated [9] for $N_f = N_c$ and $N_f = N_c + 1$ where the results are in agreement with those in [5]. Unfortunately, the GKA equations are quite complicated for $N_f \geq N_c + 2$ flavors. We will show below how to simplify the GKA equations when $N_f = N_c + 2$.

We briefly recap the derivation of the equations for $\mathcal{W}_{\text{eff}}$ from the GKA equations [9]. The strategy is the same as in the SU(2) case discussed in the last section: start with the tree level superpotential

$$\mathcal{W}_{\text{tree}} = m^i_j(\hat{M}^j_i - M^j_i) + b^{ij}(\hat{B}_{ij} - B_{ij}) + \widetilde{b}_{ij}(\hat{\tilde{B}}^{ij} - \tilde{B}^{ij}) \tag{3.4}$$

where

$$m^i_j = -\frac{\partial \mathcal{W}_{\text{eff}}}{\partial M^j_i}, \qquad b^{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\text{eff}}}{\partial B_{ij}}, \qquad \widetilde{b}_{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\text{eff}}}{\partial \tilde{B}^{ij}}, \tag{3.5}$$

are Lagrange multipliers enforcing that M^j_i, B_{ij} and $\tilde{B}_{ij}$ be the vevs of the meson, baryon, and anti-baryon operators, respectively. There is no need to introduce a Lagrange multiplier for S because we are considering points away from the origin and S is massive for these points. We get two relations among the Lagrange multipliers by taking F_r to be the quark or antiquark field in the GKA equation (2.4), and two more by taking it to be proportional to $\epsilon^{ijk\ell_2\cdots\ell_{N_c}}\epsilon_{aa_2\cdots a_{N_c}}\tilde{Q}^{a_2}_{\ell_2}\cdots\tilde{Q}^{a_{N_c}}_{\ell_{N_c}}$, and similarly with the Q's. The resulting GKA equations are

$$\begin{aligned} M^i_k m^k_j &= (S + b^{k\ell}B_{\ell k})\delta^i_j - 2b^{ik}B_{kj}, \\ m^i_k M^k_j &= (S + \widetilde{b}_{k\ell}\tilde{B}^{\ell k})\delta^i_j - 2\widetilde{b}^{ik}\tilde{B}_{kj}, \\ m^{[j}_i \tilde{B}^{k\ell]} &= 2b^{hg} {M^{-1}}^{[j}_{g} {M^{-1}}^{k}_{h} {M^{-1}}^{\ell]}_{i} \det M, \\ m^i_{[j} B_{k\ell]} &= 2\widetilde{b}_{hg} {M^{-1}}^{g}_{[j} {M^{-1}}^{h}_{k} {M^{-1}}^{i}_{\ell]} \det M. \end{aligned} \tag{3.6}$$

Note that the right sides of the last two equations, though they are written using M^{-1}, are actually polynomial in M.

Now, the global flavor symmetry implies[3] that

$$\mathcal{W}_{\rm eff} = S\, f(X, S^{-2} \det M), \qquad \text{where} \ \ X := \frac{M^T B M \tilde{B}}{\det M}, \tag{3.7}$$

and we are using matrix notation for the meson and baryon fields. The first two equations in (3.6) then imply

$$\mathcal{W}_{\rm eff} = S\, W(X) - S \ln \det M + 2S(\ln S - 1), \tag{3.8}$$

where W is to be determined.

The GKA equations (3.6) imply a matrix differential equation for $W(X)$ as follows. Contract the $ijk\ell$ indices in the last equation in (3.6) with three M's, giving $2(2 - N_f)\widetilde{b} \det M = \mathrm{tr}(mM) M^T BM + M^T m^T M^T BM - M^T BMmM$. Substitute for Mm using the first equation to get

$$2\, \widetilde{b} \det M = - \left[S + \frac{N_c - 2}{N_c} \mathrm{tr}(bB) \right] M^T BM - \frac{4}{N_c} M^T BbBM. \tag{3.9}$$

Derivatives of (3.8) with respect to B and $\tilde{B}$ together with (3.5) imply

$$\begin{aligned} 2\, b \det M &= S\, (M\tilde{B} W_X M^T + M W_X^T \tilde{B} M^T), \\ 2\, \widetilde{b} \det M &= S\, (W_X M^T BM + M^T BM W_X^T), \end{aligned} \tag{3.10}$$

where $W_X := \partial W / \partial X$. Work at a generic point in parameter space where M, B, $\tilde{B}$ and X are all invertible matrices. This is only possible if N_f, and therefore N_c, is taken to be even, as in the discussion in the previous section.

Substitute (3.10) in (3.9) and multiply on both left and right by $\tilde{B}$ to obtain

$$\tilde{B} G(X) = -G^T(X) \tilde{B}, \tag{3.11}$$

where

$$G(X) := W_X X + \frac{1}{2} \left[1 + \frac{N_c - 2}{N_c} \mathrm{tr}(W_X X) \right] X + \frac{2}{N_c} X W_X X, \tag{3.12}$$

[3] This symmetry argument is not entirely straightforward. The $\mathrm{U}(1)_B$ baryon number symmetry implies that for each B there must be an accompanying $\tilde{B}$ in each term. Since $\mathcal{W}_{\rm eff}$ is an $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ singlet, all the flavor indices must be contracted in each term. Contractions with the totally antisymmetric epsilon tensors can always be reduced to products of $\det M$ and $\mathrm{Pf}\, B \cdot \mathrm{Pf}\, \tilde{B}$. The only other way to contract indices of B and $\tilde{B}$ is with an M as $BM\tilde{B}$ (or its transpose), and since these in turn must be contracted, another factor of M must be included. There are four ways of doing this—$M^T BM\tilde{B}$ and its three cyclic permutations—but upon making a flavor singlet expression a trace must be taken, so the cyclic order does not matter. Finally, the product of Pfaffians of baryons is not independent of X and $\det M$, since $\mathrm{Pf}\, B \cdot \mathrm{Pf}\, \tilde{B} = \sqrt{\det(M^T BM\tilde{B})} / \det M$.

Alternatively, one can derive this directly from the GKA equations. Use them to deduce (3.9) and a similar relation for b, multiply these by $\tilde{B}$ and B, respectively, then substitute the second into the first. One finds that $\widetilde{b}\tilde{B}$ depends on B and $\tilde{B}$ only through X. Since $\widetilde{b} \sim \partial \mathcal{W}_{\rm eff} / \partial \tilde{B}$, it follows that the dependence of $\mathcal{W}_{\rm eff}$ on B and $\tilde{B}$ is solely through X.

Note that this symmetry argument is no longer effective when $N_f > N_c + 2$. For then B and $\tilde{B}$ have more than two indices, and the analog of X is no longer a matrix, but has $N_f - N_c - 1$ upper and $N_f - N_c - 1$ lower (antisymmetrized) indices. These objects can be contracted in many inequivalent ways to make flavor singlets. This is the source of the difficulty in integrating the GKA equations for $N_f > N_c + 2$.

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and we have used that $\tilde{B}X$ is antisymmetric. On the other hand, from the definition of X (3.7) it follows that $\tilde{B}X = X^T\tilde{B}$, which implies

$$\tilde{B}\, G(X) = G^T(X)\, \tilde{B} \tag{3.13}$$

since G is a function of X alone. (3.13) and (3.11) imply $G(X) = 0$, which, after being multiplied from the right by X^{-1}, reads

$$W_X + \frac{1}{2}\left(1 + \frac{N_c - 2}{N_c}\operatorname{tr}(W_X X)\right)\mathbb{I} + \frac{2}{N_c} X W_X = 0, \tag{3.14}$$

where $\mathbb{I}$ is the $N_f \times N_f$ identity matrix. This is the matrix differential equation for $W(X)$.

The trace of (3.14), $\operatorname{tr}(W_X X) = -[2\operatorname{tr}(W_X) + N_c + 2]/N_c$, allows us to eliminate $\operatorname{tr}(W_X X)$ from (3.14), giving $N_c W_X = (N_c + 2X)^{-1}[(N_c - 2)\operatorname{tr}(W_X) - 2]$. The trace of this equation allows us to eliminate $\operatorname{tr}(W_X)$ in turn, giving the following differential equation for $W(X)$:

$$W_X = \frac{2(N_c + 2X)^{-1}}{(N_c - 2)\operatorname{tr}((N_c + 2X)^{-1}) - N_c}. \tag{3.15}$$

When $N_c \neq 2$, we define the matrix $Y := (N_c - 2)(N_c + 2X)^{-1}$, and substitute it into (3.15) to obtain

$$\frac{\partial W}{\partial Y^j_k} Y^i_k = \frac{1}{N_c - \operatorname{tr}(Y)}\delta^i_j, \tag{3.16}$$

where we have explicitly written the indices to avoid any confusion. This differential equation is not integrable, as it is easy to check that $\partial^2 W/\partial Y^k_l \partial Y^i_j \neq \partial^2 W/\partial Y^i_j \partial Y^k_l$. This shows that the GKA equations for $\mathcal{W}_{\text{eff}}$ are not integrable for even values of $N_c > 2$.

3.1 Comparison to the SU(2) solution

For $N_c = 2$ we are integrating the same GKA equations as we did in section 2, though in terms of the baryons and anti-baryons B and $\tilde{B}$ instead of their Hodge duals $\mathcal{B}$ and $\tilde{\mathcal{B}}$. Thus the GKA equations must be integrable in this case, and, of course, give the same answer we found in section 2, namely, equation (2.15) with $N_f = 4$. However, there is a subtlety in comparing these two computations, which we will now explain. It will play an important part in our discussion of the results of integrating the Seiberg dual GKA equations in the next section.

For $N_c = 2$, (3.14) is indeed integrable, and integrates to give $W(X) = -\frac{1}{2}\operatorname{tr}\ln(1 + X)$. Integrating out S from (3.8) then gives

$$\Lambda \mathcal{W}_{\text{eff,SU(2)}} = -2\sqrt{\det M}\,\det(1 + X)^{1/4}, \tag{3.17}$$

where Λ is the strong coupling scale of the gauge theory. On the other hand, the SU(2) effective superpotential for $N_f = 4$ found in section 2 is

$$\Lambda \mathcal{W}_{\text{eff,SU(2)}} = -2\sqrt{\det M}\,\det(1 + M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B})^{1/4}. \tag{3.18}$$

Since the SU(2) baryon fields $\mathcal{B}$ and $\tilde{\mathcal{B}}$ as defined in (2.1) are Hodge-dual to the SU(N_c) baryons B, $\tilde{B}$ defined in (3.1), and since $\operatorname{rank}(M) = N_f = 4$, it follows that

$$M^{-T}\tilde{\mathcal{B}}M^{-1}\mathcal{B} = \frac{1}{2}\operatorname{tr}(X) - X, \tag{3.19}$$

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so that the effective superpotential reads

$$\Lambda \mathcal{W}_{\text{eff,SU}(2)} = -2\sqrt{\det M}\,\det(1+\tfrac{1}{2}\operatorname{tr}(X)-X)^{1/4}. \tag{3.20}$$

Apparently the two answers, (3.17) and (3.20), do not agree for general X.

The resolution of this paradox is that X is *not* a general 4×4 complex matrix, but satisfies some constraints by virtue of its definition (3.7). In particular, X can be thought of as the product of two antisymmetric matrices ($M^T BM$ and $\tilde{B}$). Such a matrix, though not necessarily either symmetric or antisymmetric, has only half the degrees of freedom that a general matrix, Y, of the same rank would have.

To see this, recall that an appropriate similarity transformation $G^{-1}YG$ with $G \in \mathrm{GL}(2n,\mathbb{C})$ will diagonalize Y. Thus the $2n$ eigenvalues of the general rank $2n$ matrix Y are all its $\mathrm{GL}(2n,\mathbb{C})$-invariants. More properly, a basis of $2n$ independent symmetric polynomials of these eigenvalues generates all invariants. This basis can conveniently be taken to be $\operatorname{tr}(Y^p)$ for $p=1,\ldots,2n$. If, on the other hand, $X=AB$ is the product of two antisymmetric matrices A and B, then a $\mathrm{GL}(2n,\mathbb{C})$ transformation acts as $G^{-1}XG = G^{-1}ABG = G^{-1}AG^{-T}G^TBG$. We can always choose a $G=G_0$ so that $G_0^TBG_0 = J$ where $J := \mathbb{I}_n \otimes i\sigma_2$ is the "unit" skew-diagonal matrix. This condition does not fix G, since if $H\in \mathrm{Sp}(2n,\mathbb{C})$ (i.e., $H \in \mathrm{GL}(2n,\mathbb{C})$ satisfies $H^TJH = J$), then $G=G_0H$ will also give $G^TBG = J$. For such G we have $G^{-1}XG = H^{-1}A'H^{-T}J$ where A' is the antisymmetric matrix $A' = G_0^{-1}AG_0^{-T}$. Now, a Gram-Schmidt othogonalization argument but with respect to the skew product defined by J shows that H can always be chosen to bring $H^{-1}A'H^{-T}$ to skew diagonal form $H^{-1}A'H^{-T} = \operatorname{diag}\{a_1,\ldots,a_n\}\otimes i\sigma_2$. Thus X has only n independent (double) eigenvalues, and a basis of generators of the $\mathrm{GL}(2n,\mathbb{C})$-invariants of X can be taken to be $\operatorname{tr}(X^p)$ for $p=1,\ldots,n$. This is half the number of independent invariants of the general rank $2n$ matrix Y, and implies, in particular, that $\operatorname{tr}(X^p)$ for $p>n$ satisfy additional identities allowing them to be expressed in terms of products of traces with $p\le n$.

For example, for $N_f = 4$ ($n=2$) the new independent cubic and quartic identities are easily found to be

$$\begin{aligned}
0 &= 8\operatorname{tr}X^3 - 6\operatorname{tr}X^2\operatorname{tr}X + (\operatorname{tr}X)^3,\\
0 &= 8\operatorname{tr}X^4 - 4\operatorname{tr}X^3\operatorname{tr}X - 2(\operatorname{tr}X^2)^2 + \operatorname{tr}X^2(\operatorname{tr}X)^2.
\end{aligned} \tag{3.21}$$

When $N_f=4$, the identities (3.21) are easily checked to imply that $\operatorname{tr}(X^p) = \operatorname{tr}(X'^p)$ for $p=1,\ldots,4$, where $X' := \frac{1}{2}\operatorname{tr}(X) - X$. Thus all invariants made from X' are the same as those for X, and in particular, the two forms of the effective superpotential (3.17) and (3.20) are equivalent.

4. $\mathcal{W}_{\text{eff}}$ for $\mathbf{N_f = N_c + 2}$ superQCD: Seiberg dual analysis

The IR-equivalent Seiberg dual description [6] of $\mathrm{SU}(N_c)$ superQCD with $N_f = N_c+2$ is $\mathrm{SU}(2)$ superQCD with N_f (dual) quarks q_i^a in the fundamental and N_f (dual) anti-quarks $\tilde{q}_a^i$ in the anti-fundamental, and a set of gauge singlets $\hat{\mathcal{M}}_j^i$ coupled through the superpotential

$$\mathcal{W} = \tilde{q}^i_a q^a_j \hat{\mathcal{M}}^j_i, \tag{4.1}$$

where $i = 1, \ldots, N_f$ and $a = 1, 2$ are flavor and color indices, respectively. This superpotential breaks the global symmetry of the dual theory down to $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$. The dual meson, baryons, and glueball are defined to be

$$\hat{\mathcal{N}}^i_j := \tilde{q}^i_a q^a_j, \qquad \hat{\mathcal{B}}_{ij} := \epsilon_{ab} q^a_i q^b_j, \qquad \hat{\tilde{\mathcal{B}}}^{ij} := \epsilon^{ab} \tilde{q}^i_a \tilde{q}^j_b, \qquad \mathcal{S} := \mathrm{tr}(w^\alpha w_\alpha)/(32\pi^2). \tag{4.2}$$

This dual theory is IR free when $N_f \geq 6$ ($N_c \geq 4$), and there are free quarks, anti-quarks and gluons at the origin of the moduli space. The $\mathrm{U}(1)_R$ charges are $R(\mathcal{S}) = 2$, $R(\mathcal{M}) = 4/N_f$, and $R(\mathcal{N}) = R(\mathcal{B}) = R(\tilde{\mathcal{B}}) = 2N_c/N_f$. The chiral ring of the dual theory $(\mathcal{S}, \mathcal{M}, \mathcal{B}, \tilde{\mathcal{B}})$ is related to that of the direct theory $(S, M, B, \tilde{B})$ by [1]

$$S = -\mathcal{S}, \qquad M = \mu \mathcal{M}, \qquad B = i\mu^{-1}\Lambda^{N_f-3}\mathcal{B}, \qquad \tilde{B} = i\mu^{-1}\Lambda^{N_f-3}\tilde{\mathcal{B}}, \tag{4.3}$$

where μ is a matching scale defined by

$$\tilde{\Lambda}^{6-N_f} = \mu^{N_f}\Lambda^{6-2N_f}, \tag{4.4}$$

with Λ and $\tilde{\Lambda}$ being the dynamical scales of the direct and dual theories, respectively.

The gauge-invariant form of the classical F-term equations are

$$\mathcal{N} = \mathcal{B}\mathcal{M} = \mathcal{M}\tilde{\mathcal{B}} = 0, \tag{4.5}$$

where we are using a matrix notation for the fields. The D-term equations give

$$\mathcal{B} \wedge \mathcal{N} = \mathcal{N} \wedge \tilde{\mathcal{B}} = \mathcal{B} \wedge \mathcal{B} = \tilde{\mathcal{B}} \wedge \tilde{\mathcal{B}} = \mathcal{B} \otimes \tilde{\mathcal{B}} - \mathcal{N} \wedge \mathcal{N} = 0, \tag{4.6}$$

which are the same as the constraints (2.2) of SU(2) superQCD discussed in section 2 with the substitution $M \to \mathcal{N}$. The space of solutions to the classical constraints (4.5–4.6) has two branches: either $\mathcal{N} = \mathcal{B} = \tilde{\mathcal{B}} = 0$ and $\mathrm{rank}(\mathcal{M}) > N_c$, or, up to flavor rotations,

$$\mathcal{N} = 0, \qquad \mathcal{M} = \begin{pmatrix} \mathbf{m} & & \\ & & \\ & & \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} & & \\ & & b \\ & -b & \end{pmatrix}, \qquad \tilde{\mathcal{B}} = \begin{pmatrix} & & \\ & & \tilde{b} \\ & -\tilde{b} & \end{pmatrix}, \tag{4.7}$$

where $\mathbf{m}$ is an $N_c \times N_c$ matrix and $b\tilde{b} = 0$.

This classical moduli space is clearly not the same as the moduli space (3.2–3.3) of the direct $\mathrm{SU}(N_c)$ theory. However, the dual theory classical constraints (4.5–4.7) are expected to be modified quantum mechanically by strong coupling effects. Because even though the dual theory is free at the origin of moduli space, arbitrarily small vevs for $\mathcal{M}$ destabilize the free fixed point by giving masses to the dual quarks through the superpotential coupling (4.1), so the theory flows to strong coupling for large enough rank of $\mathcal{M}$. Strong coupling effects are argued in [6] to generate the constraints (3.2) of the direct theory.

In particular, the branch of (4.5–4.7) with $\mathrm{rank}(\mathcal{M}) > N_c$ is lifted, and the $b\tilde{b} = 0$ constraint on the other branch is deformed to $b\tilde{b} = \det(\mathbf{m})$. As an example—which will be

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useful later—of how the classical constraints are modified by strong coupling effects, note that if the singlet vev $\mathcal{M}$ is given a generic value (e.g., by constraining it with a Lagrange multiplier term as we will do below) with $\mathcal{B} = \tilde{\mathcal{B}} = 0$, the superpotential (4.1) gives mass to all the dual quarks. The dual theory thus flows to SU(2) superYang-Mills in the IR with glueball vev $\mathcal{S} = (\tilde{\Lambda}^{6-N_f} \det \mathcal{M})^{1/2}$ generated by a Veneziano-Yankielowicz superpotential

$$\mathcal{W}_{\mathrm{eff,VY}} = 2\mathcal{S}\left(1 - \ln\left[\mathcal{S}\tilde{\Lambda}^{(N_f-6)/2}/\sqrt{\det \mathcal{M}}\right]\right), \tag{4.8}$$

where $\tilde{\Lambda}$ is the strong-coupling scale of the dual theory. The scale of the glueball appearing this superpotential is determined by one loop matching $\tilde{\Lambda}^6_{\mathrm{YM}} = \tilde{\Lambda}^{6-N_f} \det \mathcal{M}$. Integrating $\mathcal{S}$ out of (4.8) then generates an effective superpotential for the singlet field given by $2\tilde{\Lambda}^{(6-N_f)/2}\sqrt{\det \mathcal{M}}$, thus lifting the rank$(\mathcal{M}) = N_f$ region of the classical moduli space.

4.1 Derivation of $\mathcal{W}_{\mathrm{eff}}$

We now wish to find an exact effective superpotential for the dual theory which reproduces the strong quantum effects described in the last two paragraphs. As in previous sections we start with a tree level superpotential

$$\mathcal{W}_{\mathrm{tree}} = \hat{\mathcal{N}}^i_j \hat{\mathcal{M}}^j_i + m^i_j(\hat{\mathcal{M}}^j_i - \mathcal{M}^j_i) + b^{ij}(\hat{\mathcal{B}}_{ij} - \mathcal{B}_{ij}) + \widetilde{b}_{ij}(\hat{\tilde{\mathcal{B}}}^{ij} - \tilde{\mathcal{B}}^{ij}), \tag{4.9}$$

where

$$m^i{}_j = -\frac{\partial \mathcal{W}_{\mathrm{eff}}}{\partial \mathcal{M}^j{}_i}, \qquad b^{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\mathrm{eff}}}{\partial \mathcal{B}^{ij}}, \qquad \widetilde{b}_{ij} = -\frac{1}{2}\frac{\partial \mathcal{W}_{\mathrm{eff}}}{\partial \tilde{\mathcal{B}}^{ij}}, \tag{4.10}$$

are the by-now familiar Lagrange multipliers. We have only specified the vevs of $\hat{\mathcal{M}}$, $\hat{\mathcal{B}}$, and $\hat{\tilde{\mathcal{B}}}$, because they completely parametrize the moduli space.

Using the GKA equations (2.4) in precisely the same way as before, we find a set of equations similar to those, (2.7–2.9), found in section 2,

$$\mathcal{M}\mathcal{N} = \mathcal{S} + 2b\mathcal{B}, \qquad \mathcal{N}\mathcal{M} = \mathcal{S} + 2\tilde{\mathcal{B}}b, \qquad \tilde{\mathcal{B}}\mathcal{M}^T = -2\mathcal{N}b, \qquad \mathcal{M}^T\mathcal{B} = -2\widetilde{b}\mathcal{N}, \tag{4.11}$$

together with the equation coming from the variation of the singlet field $\mathcal{M}$:

$$\mathcal{N} = -m. \tag{4.12}$$

Eliminating $\mathcal{N}$ by plugging (4.12) into (4.11) gives a set of partial differential equations for the effective superpotential

$$-\mathcal{M}m = \mathcal{S} + 2b\mathcal{B}, \qquad -m\mathcal{M} = \mathcal{S} + 2\tilde{\mathcal{B}}\widetilde{b}, \qquad \tilde{\mathcal{B}}\mathcal{M}^T = 2mb, \qquad \mathcal{M}^T\mathcal{B} = 2\widetilde{b}m. \tag{4.13}$$

To solve for the effective superpotential we multiply the third equation in (4.13) from the right by $\mathcal{B}$ and from the left by $\mathcal{M}$ to obtain $\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B} = 2\mathcal{M}mb\mathcal{B} = -2(\mathcal{S}\mathbb{I} + 2b\mathcal{B})b\mathcal{B}$, where in the second equality we used the first equation in (4.13) to eliminate $\mathcal{M}m$. Solving this quadratic equation for $b\mathcal{B}$, we find

$$\begin{aligned} b\mathcal{B} &= \frac{1}{4}\left(-\mathcal{S} + \sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}}\right), \qquad \mathcal{M}m = \frac{1}{2}\left(-\mathcal{S} - \sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}}\right), \\ \tilde{\mathcal{B}}\widetilde{b} &= \frac{1}{4}\left(-\mathcal{S} + \sqrt{\mathcal{S}^2 - 4\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}\mathcal{M}}\right), \qquad m\mathcal{M} = \frac{1}{2}\left(-\mathcal{S} - \sqrt{\mathcal{S}^2 - 4\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}\mathcal{M}}\right), \end{aligned} \tag{4.14}$$

where the second line comes from solving similar quadratic equations for $\tilde{\mathcal{B}}\widetilde{b}$ and $m\mathcal{M}$.

The matrix square roots in the above expressions need some explanation. In order to make sense of them, consider a region in the configuration space where the magnitude of each element in the matrix $\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}$ is much smaller than $\mathcal{S}^2$. We can then expand the the square root as a power series, $\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} = \mathcal{S}\sqrt{\mathbb{I}}(\mathbb{I} - 2\mathcal{S}^{-2}\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B} + \cdots)$, and by analytic continuation we extend the result to include all points on the parameter space. However, for this to be a definition of the square root, we still need to determine $\sqrt{\mathbb{I}}$. In general $\sqrt{\mathbb{I}} = 2P - \mathbb{I}$ where P can be any projection matrix ($P^2 = P$). Which P should we use? The following argument shows that we have to take $\sqrt{\mathbb{I}} = \pm\mathbb{I}$. As we saw in the discussion surrounding (4.8), we expect to generate a non-zero $\mathcal{S}$ at points in the parameter space where $\mathcal{M} \neq 0$ but $\mathcal{B} = \tilde{\mathcal{B}} = 0$, so this is a suitable region to evaluate the square roots. At such points (4.13) implies that $m\mathcal{M} = \mathcal{M}m = -\mathcal{S}$ and $\tilde{\mathcal{B}}\mathcal{M}^T = \mathcal{M}^T\mathcal{B} = 0$. But this is only consistent with (4.14) if $\sqrt{\mathcal{S}^2\mathbb{I}} = \mathcal{S}\mathbb{I}$ implying that $\sqrt{\mathbb{I}} = \pm\mathbb{I}$.

It is straightforward to integrate (4.14) for the effective superpotential to get

$$\begin{aligned}\mathcal{W}_{\text{eff}} &= \frac{\mathcal{S}}{4}\ln\det\left(\frac{\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} - \mathcal{S}}{\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} + \mathcal{S}}\right) - \frac{\mathcal{S}}{4}\ln\det(\tilde{\Lambda}^{-6}\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}) + \mathcal{S}\ln\det(\tilde{\Lambda}^{-1}\mathcal{M}) \\ &\quad + \frac{1}{2}\operatorname{tr}\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} + \frac{1}{2}(4 - N_f)\mathcal{S}\left[\alpha - \ln(\mathcal{S}/\tilde{\Lambda}^3)\right]. \qquad (4.15)\end{aligned}$$

We used the $\mathrm{U}(1)_R$ symmetry to fix the $\mathcal{S}$ dependence up to an undetermined integration constant α. Evaluating $\mathcal{W}_{\text{eff}}$ at $\mathcal{B} = \tilde{\mathcal{B}} = 0$ gives, after a somewhat delicate cancellation,

$$\mathcal{W}_{\text{eff}}(\mathcal{B} = \tilde{\mathcal{B}} = 0) = 2\mathcal{S}\left(\alpha + (1 - \alpha)\frac{N_f}{4} - \ln\left[\mathcal{S}\tilde{\Lambda}^{(N_f/2)-3}/\sqrt{\det\mathcal{M}}\right]\right). \qquad (4.16)$$

Comparing to the answer (4.8) expected from the strong coupling analysis, fixes $\alpha = 1$. Note that this limiting form (4.16) is already a check that the effective superpotential (4.15) is consistent with the quantum modified constraints. In particular the $\mathcal{S}\ln\det\mathcal{M}$ term in (4.16) serves to lift the whole rank$\mathcal{M} = N_f$ region of the classical moduli space (4.5–4.7), in accordance with the expected quantum constraints (3.2–3.3).

So, our final result for the effective superpotential for the Seiberg dual theory is

$$\begin{aligned}\mathcal{W}_{\text{eff}} &= \frac{\mathcal{S}}{4}\ln\det\left(\frac{\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} - \mathcal{S}}{\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} + \mathcal{S}}\right) - \frac{\mathcal{S}}{4}\ln\det\left(\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}\right) + \mathcal{S}\ln\det\mathcal{M} \\ &\quad + \frac{1}{2}\operatorname{tr}\sqrt{\mathcal{S}^2 - 4\mathcal{M}\tilde{\mathcal{B}}\mathcal{M}^T\mathcal{B}} + (4 - N_f)\frac{\mathcal{S}}{2}(1 - \ln\mathcal{S}) + (6 - N_f)\mathcal{S}\ln\tilde{\Lambda}. \qquad (4.17)\end{aligned}$$

Using the chiral ring mappings (4.3), this can be expressed in terms of the fields of the direct theory. Note that since $S = -\mathcal{S}$, the definition of the branch of the square root made above, $\sqrt{\mathcal{S}^2\mathbb{I}} = \mathcal{S}\mathbb{I}$, now becomes $\sqrt{S^2\mathbb{I}} = -S\mathbb{I}$. We make this minus sign explicit by changing the signs of all square roots and keeping the convention $\sqrt{S^2\mathbb{I}} = +S\mathbb{I}$. We then find

$$w_{\text{eff}} = \frac{s}{4}\ln\left[\det\left(\frac{\sqrt{s^2 + 4x} + s}{\sqrt{s^2 + 4x} - s}\right)\frac{\det x}{\det^4 M}\right] - \frac{1}{2}\operatorname{tr}\sqrt{s^2 + 4x} + (N_f - 4)\frac{s}{2}\left[1 - \ln s\right], \quad (4.18)$$

where we have defined the shorthands

$$w_{\text{eff}} := \Lambda^{N_f-3}\mathcal{W}_{\text{eff}}, \qquad s := \Lambda^{N_f-3}S, \qquad x := M\tilde{B}M^T B, \tag{4.19}$$

to avoid having to write factors of Λ. This is the effective superpotential for SU(N_c) superQCD with $N_f = N_c + 2$.

4.2 Integrating out the glueball field

Away from the origin of moduli space, the glueball s is expected to be massive. Solving its equation of motion, $\partial w_{\text{eff}}/\partial s = 0$, gives $s = s_*(M, B, \tilde{B})$ where s_* is defined implicitly by

$$s_*^{2(N_f-4)} = \det\left(\frac{\sqrt{s_*^2+4x}+s_*}{\sqrt{s_*^2+4x}-s_*}\right)\frac{\det x}{\det^4 M}. \tag{4.20}$$

Substituting this in w_{eff} gives the effective superpotential as a function of meson and baryons only,

$$w_{\text{eff}}|_{s_*} = -2s_*\left[1+\frac{1}{4}\operatorname{tr}\left(\sqrt{1+4\frac{x}{s_*^2}}-1\right)\right]. \tag{4.21}$$

The relation (4.20) gives s_* as a complicated function of M, B, and $\tilde{B}$, which makes it difficult to deduce the equations of motion for these fields from w_{eff}.

Before solving (4.20) for s_*, we can extract some of the constraints that M, B, and $\tilde{B}$ must satisfy on the moduli space. Since w_{eff} is singular, as shown in [7] it must be regularized in order to extract its physical predictions. The idea behind the regularization is to deform $\mathcal{W}_{\text{eff}}$ by introducing some regularizing parameters μ, β, $\tilde{\beta}$ as follows

$$w_{\text{eff}} \to w_{\text{eff}}^{\mu,\beta,\tilde{\beta}} = w_{\text{eff}} + \mu^i_j M^j_i + \beta^{ij}B_{ij} + \tilde{\beta}_{ij}\tilde{B}^{ij}. \tag{4.22}$$

Generic small values of μ, β, and $\tilde{\beta}$, will fix M, B, and $\tilde{B}$ to some values which will be shifted from the moduli space (the extrema of w_{eff}) by positive powers of the regularizing parameters. Thus as $\mu, \beta, \tilde{\beta} \to 0$, $\{M, B, \tilde{B}\}$ will approach some point on the moduli space. However, the specific point reached on the moduli space will depend on how the regularizing parameters scale to zero: as different orders of limits of $\mu, \beta, \tilde{\beta} \to 0$ are taken, the whole moduli space will be scanned. (Note that some vanishing limits of the regularizers can also send M, B, or $\tilde{B}$ to infinity; this is unavoidable since the moduli space itself stretches off to infinity.)

This regularization procedure should be compared with the trick we used of introducing Lagrange multipliers (4.9) to derive equations for w_{eff} from the GKA equations. With the replacement of the Lagrange multipliers with the regularizing parameters, $\{m, b, \widetilde{b}\} \to \{\mu, \beta, \tilde{\beta}\}$, the identification of the Lagrange multipliers as derivatives of w_{eff}, (4.10), is now interpreted as the regularized equations of motion. Thus, extremizing $w_{\text{eff}}^{\mu,\beta,\tilde{\beta}}$ using (4.18) gives

$$\begin{aligned} 4\beta B &= s - \sqrt{s^2+4x}, \qquad & 2M\mu &= s + \sqrt{s^2+4x}, \\ 4\tilde{B}\tilde{\beta} &= s - \sqrt{s^2+4y}, \qquad & 2\mu M &= s + \sqrt{s^2+4y}, \end{aligned} \tag{4.23}$$

where $y := M^{-1}xM$. (4.23) is equivalent to (4.14), the original differential equations—with the operator mappings (4.3)—we integrated to get w_{eff} in the first place. Then, reversing the manipulations which led from (4.13) to (4.14) gives

$$M\mu = s - 2\beta B, \quad \mu M = s - 2\tilde{B}\tilde{\beta}, \quad \tilde{B}M^T = -2\mu\beta, \quad M^T B = -2\tilde{\beta}\mu. \tag{4.24}$$

These show that independent of how $\mu, \beta, \tilde{\beta} \to 0$, we always end up with $\tilde{B}M^T = M^T B = 0$. This implies in particular that $x = -4\beta\mu^T\tilde{\beta}\mu$, and so also always vanishes with the regularizing parameters. Also, if as $\mu, \beta, \tilde{\beta} \to 0$ M, B, and $\tilde{B}$ remain finite, then $s \to 0$ as well.

This shows that the extrema of the superpotential (4.18) satisfy the constraints $\tilde{B}M^T = M^T B = s = 0$, which are, indeed, part of the constraints (3.2) describing the $N_f = N_c + 2$ superQCD moduli space. The remaining constraints should also follow from the effective superpotential. However, they are much harder to derive, as they require solving for s_* in (4.20). Now the argument of the last paragraph shows that $s_* = 0$, but, because of the singular nature of the superpotential, it is incorrect to simply plug this value into (4.21) to find w_{eff}. Instead, s_* should be found for generic M, B, and $\tilde{B}$, and then the extrema of the resulting effective superpotential can be analyzed by regularizing it, as above.

The equation (4.20) can be solved systematically for s_* by assuming that (all the eigenvalues of) $\xi := x/s_*^2 \ll 1$ and expanding in powers of this parameter. The leading order solution is $s_*^2 = \det M$, which can be checked to be consistent with the assumption that $||\xi|| \ll 1$. Change variables to

$$\sigma^2 := \frac{s_*^2}{\det M}, \qquad X := \frac{x}{\det M}, \tag{4.25}$$

so that (4.20) and (4.21) become

$$\sigma^{2(N_f-4)} = \det X \, \det\left(\frac{\sqrt{\sigma^2+4X}+\sigma}{\sqrt{\sigma^2+4X}-\sigma}\right) = 2^{-2N_f} \det\left(\sqrt{\sigma^2+4X}+\sigma\right)^2, \tag{4.26}$$

$$w_{\text{eff}} = -2\sqrt{\det M}\left[\sigma + \frac{1}{4}\operatorname{tr}\left(\sqrt{\sigma^2+4X}-\sigma\right)\right]. \tag{4.27}$$

Expand the right side of (4.26) in powers of X/σ and solve it consistently order-by-order in a power series expansion $\sigma^2 = 1 + \cdots$ to get

$$\begin{aligned}\sigma^2 = {} & 1 - \frac{\operatorname{tr} X}{2} - \frac{(\operatorname{tr} X)^2}{8} + \frac{3\operatorname{tr}(X^2)}{4} - \frac{(\operatorname{tr} X)^3}{12} + \frac{3\operatorname{tr} X \operatorname{tr}(X^2)}{4} - \frac{5\operatorname{tr}(X^3)}{3} \\ & - \frac{9(\operatorname{tr} X)^4}{128} + \frac{27(\operatorname{tr} X)^2\operatorname{tr}(X^2)}{32} - \frac{27\operatorname{tr}(X^2)^2}{32} - \frac{5\operatorname{tr} X\operatorname{tr}(X^3)}{2} + \frac{35\operatorname{tr}(X^4)}{8} + \mathcal{O}(X^5).\end{aligned} \tag{4.28}$$

Plugging this into (4.27) gives

$$\begin{aligned}w_{\text{eff}} = {} & -2\sqrt{\det M}\Big[1 + \frac{\operatorname{tr} X}{4} + \frac{(\operatorname{tr} X)^2}{32} - \frac{\operatorname{tr}(X^2)}{8} + \frac{5(\operatorname{tr} X)^3}{384} - \frac{3\operatorname{tr} X\operatorname{tr}(X^2)}{32} + \frac{\operatorname{tr}(X^3)}{6} \\ & + \frac{49(\operatorname{tr} X)^4}{6144} - \frac{21(\operatorname{tr} X)^2\operatorname{tr}(X^2)}{256} + \frac{9\operatorname{tr}(X^2)^2}{128} + \frac{5\operatorname{tr} X\operatorname{tr}(X^3)}{24} - \frac{5\operatorname{tr}(X^4)}{16} \\ & + \mathcal{O}(X^5)\Big].\end{aligned} \tag{4.29}$$

This is the answer, to order X^4, for the effective superpotential for $N_f = N_c+2$ superQCD found by integrating the Seiberg dual GKA equations and integrating out the glueball.

Alternatively, we can derive a differential equation satisfied by $w_{\rm eff}$ as a result of integrating out s. Two of the equations of motion that we originally integrated to get the superpotential (4.23) become, when written in terms of σ and X,

$$\begin{aligned} 4\beta B &= \sqrt{\det M}\left(\sigma - \sqrt{\sigma^2+4X}\right), \\ 2M\mu &= \sqrt{\det M}\left(\sigma + \sqrt{\sigma^2+4X}\right). \end{aligned} \tag{4.30}$$

On the other hand, upon integrating out s, we have seen that $w_{\rm eff}$ takes the form

$$w_{\rm eff} = \sqrt{\det M}\, f(X) \tag{4.31}$$

for some function f. (This form just follows from the symmetries.) Two of the equations of motion following from this form of $w_{\rm eff}$ are

$$\begin{aligned} 4\beta B &= \sqrt{\det M}\left(\quad 4Xf' \ \right), \\ 2M\mu &= \sqrt{\det M}\left(-4Xf' - f + 2\,{\rm tr}(Xf')\right), \end{aligned} \tag{4.32}$$

where f' is the matrix derivative df/dX.

Equating and adding (4.30) and (4.32) gives

$$\sigma = -\frac{1}{2}f + {\rm tr}(Xf'), \tag{4.33}$$

while equating and multiplying (4.30) and (4.32) gives

$$1 = f'\left(f - 2\,{\rm tr}(Xf') + 4Xf'\right). \tag{4.34}$$

This is a nonlinear first order (matrix) differential equation for $w_{\rm eff}$ with the glueball integrated out. We do not know how to integrate this equation in closed form when ${\rm rank}(X) > 1$. But it is straightforward to check that a series expansion of the solution to (4.34) with boundary condition $f(0) = -2$ reproduces (4.29).

It may be clarifying to note that (4.34) has a one-parameter family of solutions. For if $f(X)$ is a solution, then so is

$$f_{(a)}(X) := af(a^{-2}X) \tag{4.35}$$

for any $a \in \mathbb{C}^*$. By (4.33) this implies that $\sigma(X)$ changes to $a\sigma(a^{-2}X)$; but it is easy to check that the σ equation of motion (4.26) does *not* have this symmetry. The algebraic equation (4.26) determining σ has more information than the differential equation (4.34), and so picks out a single instance of the family of solutions (4.35), namely the one obeying the boundary condition $f(0) = -2$. (For example, $-2\,{\rm tr}(\sqrt{2X})$ solves (4.34) but does not satisfy (4.26), so is not a physical solution.)

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4.3 Comparing to the direct result when $\mathbf{N_f} = 4$

Is (4.29) the correct superpotential? A basic check is to see whether its extrema (computed by appropriately regularizing, as explained above) reproduce the moduli space given by the constraints (3.2). This seems a very difficult check to perform since we do not have a closed analytic form for $w_{\rm eff}$.

However, there is one case where we can carry out a non-trivial check. When $N_f = 4$, $N_c = 2$, so in this case we should reproduce the superpotential (3.17) found in sections 2 and 3 for the SU(2) theory. Expanding (3.17) in powers of X, we find

$$\begin{aligned}\Lambda \mathcal{W}_{\rm eff,SU(2)} = -2\sqrt{\det M}\Big[1 &+ \frac{\operatorname{tr} X}{4} + \frac{(\operatorname{tr} X)^2}{32} - \frac{\operatorname{tr}(X^2)}{8} + \frac{(\operatorname{tr} X)^3}{384} - \frac{\operatorname{tr} X \operatorname{tr}(X^2)}{32} + \frac{\operatorname{tr}(X^3)}{12} \\ &+ \frac{(\operatorname{tr} X)^4}{6144} - \frac{(\operatorname{tr} X)^2 \operatorname{tr}(X^2)}{256} + \frac{\operatorname{tr}(X^2)^2}{128} + \frac{\operatorname{tr} X \operatorname{tr}(X^3)}{48} - \frac{\operatorname{tr}(X^4)}{16} \\ &+ \mathcal{O}(X^5)\Big]. \end{aligned} \tag{4.36}$$

Though it does not coincide with the expansion (4.29) starting at order X^3, we must bear in mind the identities (3.21) that traces of powers of X satisfy starting at cubic order. One finds that the difference between (4.36) and (4.29) is proportional to these identities, and so vanishes. Thus the effective superpotential found by integrating the dual GKA equations matches the correct result at least to quartic order in X.

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